Chapter-1 Real Numbers

→ Lemma are the proved statements which need to be prove another statement.

✱ Euclid's Division Lemma
For any two +ve integer a & b, there exists two unique whole number q & r such that

$\boxed{a = bq + r}$

i.e. Dividend = Divisor × quotient + remainder

- Questions -

1. A no when divided by 61 gives 27 as quotient & 32 as remainder.

⇒ here, $b = 61$, $q = 27$ & $r = 32$

⇒ $61 \times 27 + 32$

⇒ 1679.

2. Show that every +ve integer is either even or odd.

⇒ Let a be any +ve integer divided by 2, quotient "q" & remainder "r" then using Euclid Division Lemma

$a = 2q + r$ where $0 \leq r < 2$

$r = 0, 1$

Case I - when $r = 0$

⇒ $a = 2q$, which is even.

Case II - when $r = 1$

⇒ $a = 2q + 1$, which is Odd.

3. Show that any +ve odd integer is of the form $4m+1$ or $4m+3$

⇒ Let a be an arbitary integer divided by 4, quotient 'm', remainder 'r'.

⇒ $a = 4m + r$ where $0 \le r < 4$

$r = 0, 1, 2, 3$

Case I when $r = 0$

⇒ $a = 4m$

Case II - when $r = 1$

⇒ $a = 4m + 1$

Case III - when $r = 2$

$a = 4m + 2$

Case IV - when $r = 3$

⇒ $a = 4m + 3$

4. Prove that if x & y both are odd +ve integer then x^2+y^2 is even but not divisible by 4

⇒ Let $x = 2q + 1$

$y = 2m + 1$

$\therefore x^2+y^2 = (2q+1)^2 + (2m+1)^2$

$= 4q^2 + 4q + 1 + 4m^2 + 4m + 1$

$= 4q^2 + 4m^2 + 4q + 4m + 2$

$\because x^2+y^2 = 4(q^2 + m^2 + q + m) + 2$ // not divisible by 4

5. Using Euclid division lemma show that cube of ~~the~~ any +ve integer is of the form $9m$, $9m+1$ or $9m+8$

⇒ A number 'a' divisible by 3, quotient 'q' & remainder 'r'

⇒ $a = 3q + r$ where $0 \le r < 3$

$r = 0, 1, 2$

⇒ $a^3 = (3q + r)^3$

$= 27q^3 + r^3 + 3 \times 3q \times r\,(3q + r)$

$= 27q^3 + r^3 + 9qr\,(3q + r)$

Case I - $r = 0$

⇒ $a^3 = 27q^3 + 0 + 0 + 0$

$a^3 = 27q^3$

$a^3 = 9(3q^3)$

$a^3 = 9m$ where $m = 3q^3$

Case II - $r = 1$

⇒ $a^3 = 27q^3 + 1 + 9qr\,(3q + r)$

$= 27q^3 + 1 + 27q^2r + \cancel{27}\,9qr^2$

$= 27q^3 + 1 + 27q^2r + 9q$

$= 9(3q^3 + 3q^2r + q) + 1$

$\because a^3 = 9m + 1$ where $m = 3q^3 + 3q^2r + q$

Case III - $r = 2$

⇒ $a^3 = 27q^3 + 8 + 54q^2r + 36q$

$= 27q^3 + 54q^2r + 36q + 8$

$= 9(3q^3 + 6q^2r + 4q) + 8$

$\because a^3 = 9m + 8$ where $m = 3q^3 + 6q^2r + 4q$

6. If d is the HCF of 56 & 72 then find x & y satisfying $d = 56x + 72y$. Also show that x & y are not unique.

⇒ $72 = 56 \times 1 + 16$ —— (i)

$56 = 16 \times 3 + 8$ —— (ii)

$16 = 8 \times 2 + 0$ —— (iii)

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$\Rightarrow$ 8 = 56 − 48
= 56 − 16×3 (substituting value of 48 from eq ⓘ)
= 56 − (72 − 56)×3 (substituting value of 16 from eq ⓘ)
= 56 − 72×3 + 56×3
= 56×4 − 72×3
= 56×4 + 72×(−3)
= x = 4, y = −3

$\Rightarrow$ 56×4 + 72×(−3) + (56×72 − 56×72)
$\Rightarrow$ 56×(76) + 72 (−3 − 56)
$\Rightarrow$ 56×(76) + 72×(−59)
$\Rightarrow$ ∴ x = 76, y = −59

Ques If HCF of 210 & 55 is expressible in the form 210x + 55y then, find x & y
$\Rightarrow$ Finding HCF of 210 & 55, we get 5
210 = 55×3 + 45 —— (i)
55 = 45×1 + 10 —— (ii)
45 = 10×4 + 5 —— (iii)
10 = 5×2 + 0 —— (iv)
From eq (iii)
5 $\Rightarrow$ 45 − 10×4
$\Rightarrow$ 45 − {55 − 45}×4
$\Rightarrow$ 45 − 55×4 + 45×4
$\Rightarrow$ 45×5 − 55×4
$\Rightarrow$ (210 − 55×3)5 − 55×4
$\Rightarrow$ 210×5 − 55×15 − 55×4
$\Rightarrow$ 210×5 − 55×(−19)
$\Rightarrow$ x = 5, y = (−19)

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✱ Fundamental Theorem of Arithmetic

- Every composite number can be expressed as the product of primes & this factorisation is unique.

Ques Check whether 6^n, $n \in N$ can end with the digit 0 where n is a natural number.
$\Rightarrow$ 6 = 2×3
6 has prime factor 2 & 3
Using fundamental theorem of Arithmetic prime factorization of composite number is unique.
∴ 6^n never ends with digit zero

Ques Explain why following are composite numbers :-

i) 7×11×13 + 13
$\Rightarrow$ 13(7×11 + 1)
$\Rightarrow$ 13×78
$\Rightarrow$ ∴ it is a composite no.

ii) 5(7×6×4×3×2 + 1)
$\Rightarrow$ 5 × 1009
$\Rightarrow$ So it's a composite no.

Ques Find HCF & LCM by prime factorization method of 92 & 510
$\Rightarrow$ 92 = 2×2×23
510 = 2×3×5×17
HCF = 2
LCM = 2×2×3×23×5×17
= 23460

Ques HCF (306, 657) = 9
LCM (306, 657) = ?

⇒ Product of two number = HCF × LCM

$306 \times 657 = 9 \times LCM$

$LCM = \dfrac{\overset{102}{\cancel{306}} \times \overset{\cancel{34}}{657}}{\cancel{9}\,\cancel{3}}$

$= 22338$

Ques Prove that there are infinite no of primes.

⇒ Let $p_1, p_2, p_3, \ldots, p_n$ are n finite no of prime where $p_1 < p_2 < p_3 \ldots p_n$

let $x = 1 + (p_1 p_2 p_3 \ldots p_n)$

so, x is not divisible by any prime of $p_1, p_2, \ldots, p_n$

∴ x is a prime itself. But it contradicts our supposition so there are infinite numbers of the prime.

Ques Show that 12^n can never end with digit 5 where $n \in N$.

⇒ $12 = 2 \times 2 \times 3$

12 has a factor $2 \times 2 \times 3$

12 doesn't have 5 as a factor & we know that prime factorization of a composite no is unique hence it never

Ques Find the greatest no of 6 digits exactly divisible by 24, 15, 36.

⇒

2	24, 15, 36
2	12, 15, 18
2	6, 15, 9
3	3, 15, 9
3	1, 5, 3
5	1, 5, 1

= 360

greatest 6 digit no = 999999

$$
\begin{array}{r|l}
360 & 999999 \;(2777 \\
 & 720 \\ \hline
 & 2799 \\
 & 2520 \\ \hline
 & 2799 \\
 & 2520 \\ \hline
 & 2799 \\
 & 2520 \\ \hline
 & 279
\end{array}
$$

On dividing 999999 by 360.

remainder = 279

required no = 999999 − 279

= 999720

Ques Can two numbers have 16 as their HCF & 380 as their LCM. give reason.

⇒ As we know HCF always LCM.

here,

$$
\begin{array}{r|l}
16 & 380 \;(23 \\
 & 32 \\ \hline
 & 60 \\
 & 48 \\ \hline
 & 12
\end{array}
$$

&, $380 = 16 \times 23 + 12$

so it is not possible

Ques What is the smallest no which when divided by 35, 56 & 91 leaves remainder of 7 in each.

⇒

2	35, 56, 91
2	35, 28, 91
2	35, 14, 91
5	35, 7, 91
7	7, 7, 91
13	1, 1, 91
7	1, 1, 7
	1, 1, 1

= 3640.

= 3640 + 7 = 3647

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Ques Find the smallest no when divided increased by 17 is exactly divisible by both 520 & 468

⇒

2	520, 468
2	260, 234
3	130, 117
3	65, 117
3	65, 39
5	65, 13
13	13, 13
	1, 1

$= 4680$

$= 4680 - 17$

$= 4663$

Ques Show that there is no +ve integer n for which $\sqrt{n+1} + \sqrt{n-1}$ is rational.

⇒ Let $\sqrt{n-1} + \sqrt{n+1} = \frac{p}{q}$ is rational ——— (1)

⇒ $\frac{1}{\sqrt{n-1} + \sqrt{n+1}} = \frac{q}{p}$

⇒ $\frac{\sqrt{n-1} - \sqrt{n+1}}{(\sqrt{n-1}+\sqrt{n+1})(\sqrt{n-1}+\sqrt{n+1})} = \frac{q}{p}$

⇒ $\frac{\sqrt{n-1} - \sqrt{n+1}}{(n-1)-(n+1)} = \frac{q}{p}$

⇒ $\frac{\sqrt{n-1} - \sqrt{n+1}}{n-1-n-1} = \frac{q}{p}$

⇒ $\sqrt{n-1} - \sqrt{n+1} = \frac{-2q}{p}$ ——— (ii)

Adding eq (1) & (ii)

⇒ $2\sqrt{n-1} = \frac{p}{q} + \left(\frac{-2q}{p}\right)$

⇒ $\sqrt{n-1} = \frac{p^2 - 2pq^2}{2pq}$ ⇒ $\sqrt{n-1} = \frac{p^2 - 2q^2}{2pq}$

p & q are integer ∴ $\frac{p^2-2pq}{2pq}$ is Rational

∴ $\sqrt{n-1}$ is rational i.e. $(n-1)$ is perfect square

Similarly $\sqrt{n+1}$ is rational i.e. $(n+1)$ is perfect square

diff $= (n+1)-(n-1)$

$= 2$

∴ it is not rational because the difference between 2 rational no must be more than 2

Hence it is irrational.

Ques Prove that $\sqrt{2}$ is irrational

⇒ Let $\sqrt{2}$ is rational no.

⇒ $\sqrt{2} = \frac{p}{q}$

⇒ $\frac{1}{\sqrt{2}} = \frac{q}{p}$

⇒ $\frac{\sqrt{2}}{2} = \frac{q}{p}$

⇒ $\sqrt{2} = \frac{2q}{p}$

here p & q are integers & rational so product is also rational but it contradicts our supposition. Hence, $\sqrt{2}$ is irrational.

Ques Let a & b be +ve integer, show that $\sqrt{2}$ always lies between $\frac{a}{b}$ & $\frac{a+2b}{a+b}$

⇒ Here $\frac{a}{b} < \frac{a+2b}{a+b}$ or $\frac{a}{b} > \frac{a+2b}{a+b}$

There to compare let us compute $\frac{a}{b} - \frac{a+2b}{a+b}$

here, $\Rightarrow \frac{a}{b} - \frac{a+2b}{a+b}$

$\Rightarrow \frac{a(a+b) - b(a+2b)}{b(a+b)}$

$\Rightarrow \frac{a^2 + ab - ab - 2b^2}{b(a+b)} \Rightarrow \frac{a^2 - 2b^2}{b(a+b)}$

$\therefore \frac{a}{b} - \frac{a+2b}{a+b} > 0$

$\Rightarrow \frac{a^2 - 2b^2}{b(a+b)} > 0$

$\Rightarrow a^2 - 2b^2 > 0$

$\Rightarrow a^2 > 2b^2$

$\Rightarrow a > \sqrt{2}b$ & $\frac{a}{b} - \frac{a+2b}{a+b} < 0$

$\Rightarrow \frac{a^2 - 2b^2}{b(a+b)} < 0$

$\Rightarrow a^2 - 2b^2 < 0$

$\Rightarrow a < \sqrt{2}\,b$

Thus, $\frac{a}{b} > \frac{a+2b}{a+b}$, if $a > \sqrt{2}b$ & $\frac{a}{b} < \frac{a+2b}{a+b}$, if $a < \sqrt{2}b$.

Case I - when $a > \sqrt{2}b$

$\Rightarrow \frac{a}{b} > \frac{a+2b}{a+b}$

$\Rightarrow a > \sqrt{2}b$

$\Rightarrow a^2 > 2b^2$

$\Rightarrow a^2 + a^2 > a^2 + 2b^2$ (adding a^2 both sides)

$\Rightarrow 2a^2 + 2b^2 > (a^2 + 2b^2) + 2b^2$ (adding $2b^2$ both sides)

$\Rightarrow 2(a^2+b^2) + 4ab > a^2 + 4b^2 + 4ab$

$\Rightarrow 2(a^2 + b^2 + 2ab) > a^2 + 4b^2 + 4ab$

$\Rightarrow 2(a+b)^2 > (a+2b)^2$

$\Rightarrow \sqrt{2}(a+b) > a+2b$

$\Rightarrow \sqrt{2} > \frac{a+2b}{a+b}$ —— (i)

Again,

$a > \sqrt{2}b \Rightarrow \frac{a}{b} > \sqrt{2}$ —— (ii)

From eq (i) & ii) we get.

$\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$

Similarly, in case II we can prove that

$\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$

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Ques Show that $\sqrt{p} + \sqrt{q}$ is irrational no.

$\Rightarrow$ Let $\sqrt{p} + \sqrt{q}$ is rational

$\sqrt{p} + \sqrt{q} = a$

$\sqrt{p} = a - \sqrt{q}$

Squaring both sides

$(\sqrt{p})^2 = (a - \sqrt{q})^2$

$p = a^2 - 2a\sqrt{q} + q$

$2a\sqrt{q} = a^2 + q - p$

$\sqrt{q} = \frac{a^2 + q - p}{2a}$

Hence $\sqrt{q}$ is rational. But it contradicts our supposition.

Ques - Show that $\sqrt{2}$ always lies between $\frac{a}{b}$ & $\frac{a+2b}{a+b}$

2^{nd} Method - There are 2 cases for $\frac{a+2b}{a+b}$, $\frac{a}{b}$ & $\sqrt{2}$

Case I - $\frac{a}{b} > \sqrt{2}$ i.e. $a > \sqrt{2}b$ — (i)

$$\therefore \frac{a+2b}{a+b} - \sqrt{2} = \frac{a+2b-\sqrt{2}a-\sqrt{2}b}{a+b}$$

$$= \frac{2b-\sqrt{2}b+a-\sqrt{2}a}{a+b}$$

$$= \frac{\sqrt{2}b(\cancel{\sqrt{2}}-1)-\cancel{\sqrt{2}}a+a}{a+b}$$

$$= \frac{\sqrt{2}b(\sqrt{2}-1)-a(\sqrt{2}-1)}{a+b}$$

$$= \frac{(\sqrt{2}b-a)(\sqrt{2}-1)}{a+b} \quad \text{— (ii)}$$

From eq (i) it points to negation

$\therefore \frac{a+2b}{a+b} < \sqrt{2}$ & $\frac{a+2b}{a+b} < 0$

Case II - $\frac{a}{b} < \sqrt{2}$ i.e $a < \sqrt{2}b$ — (iii)

from eq (ii)

$$\frac{a+2b}{a+b} - \sqrt{2} = \frac{(\sqrt{2}b-a)(\sqrt{2}-1)}{a+b}$$

from eq (iii)

$\frac{a+2b}{a+b} > 0$ then $\frac{a+2b}{a+b} > \sqrt{2}$

$\therefore \frac{a+2b}{a+b} > \sqrt{2} > \frac{a}{b}$